End point of first order phase transitions and sign problem in finite density lattice gauge theories

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第3回クラスター階層領域研究会
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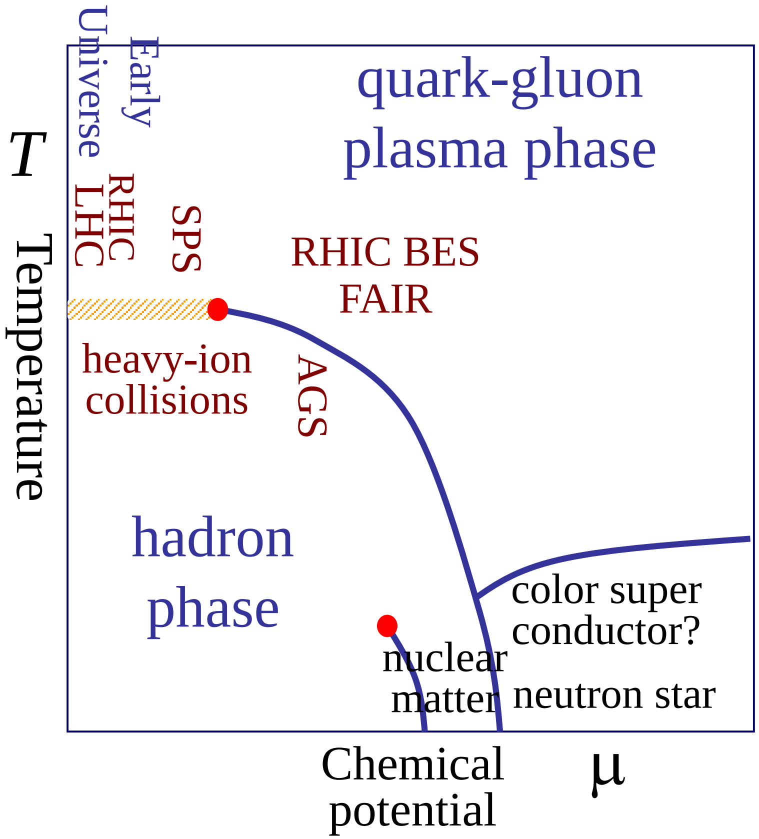
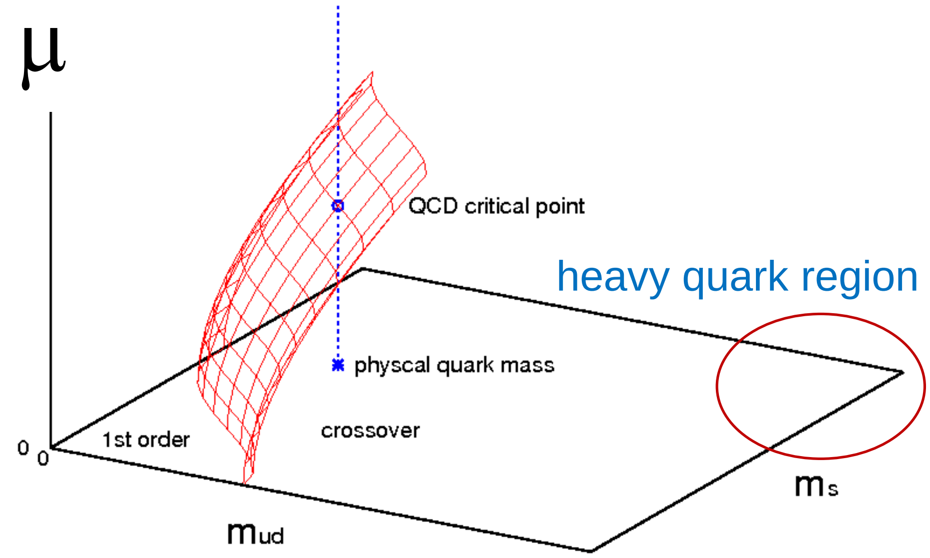
QCD phase transition at finite density



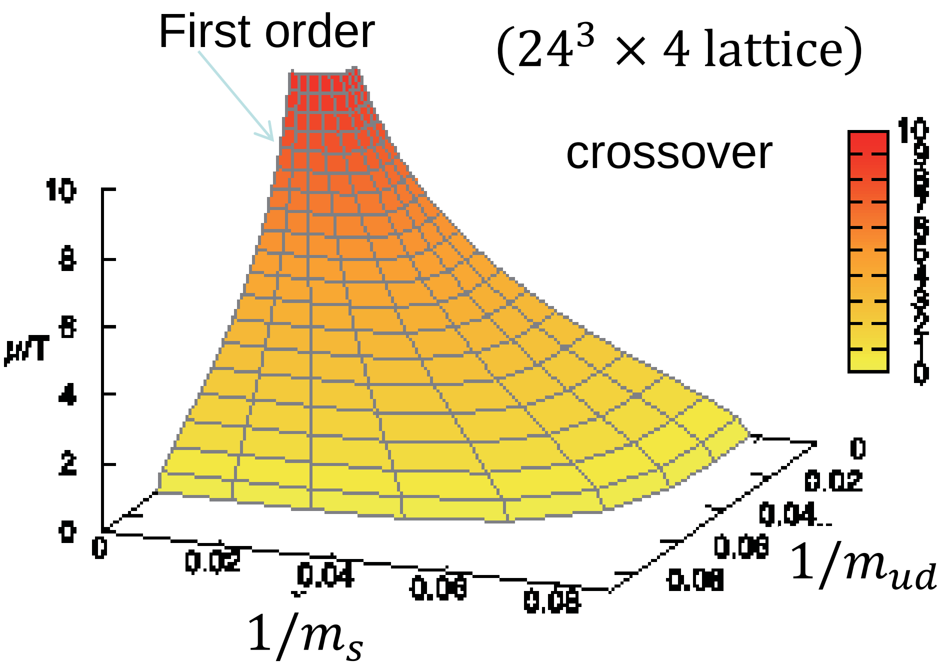


- End point of first order phase transitions in the heavy quark region. [Phys. Rev. D101, 054505 (2020)]
- Distribution function for the quark number.
  - Sign problem in the calculation of canonical partition function.
  - Center symmetry and the canonical partition function

Critical surface in the heavy quark region of (2+1)-flavor QCD

[Phys.Rev.D84, 054502(2011); Phys.Rev.D89, 034507(2014)]

Critical surface at finite density



Quenched QCD simulations + reweighting

The first order region becomes narrower as μ/T increases.

The sign problem is not seriousness.

[Phys. Rev. D101, 054505 (2020)]

Continuum limit ?

Simulations in $N_t = 6$ and 8 lattices.

Hopping parameter expansion: getting worse from $N_t = 6$.

[On going projects]

More higher order terms in the hopping parameter expansion.

Simulations with a external Polyakov loop term to avoid the overlap problem.

Sign problems in the canonical approach

- Canonical partition function: Z_C (Fugacity expansion)

$$Z_{GC}(T,\mu) = \sum_N Z_C(T,N)\exp(N\mu/T) \equiv \sum_N W(N)$$

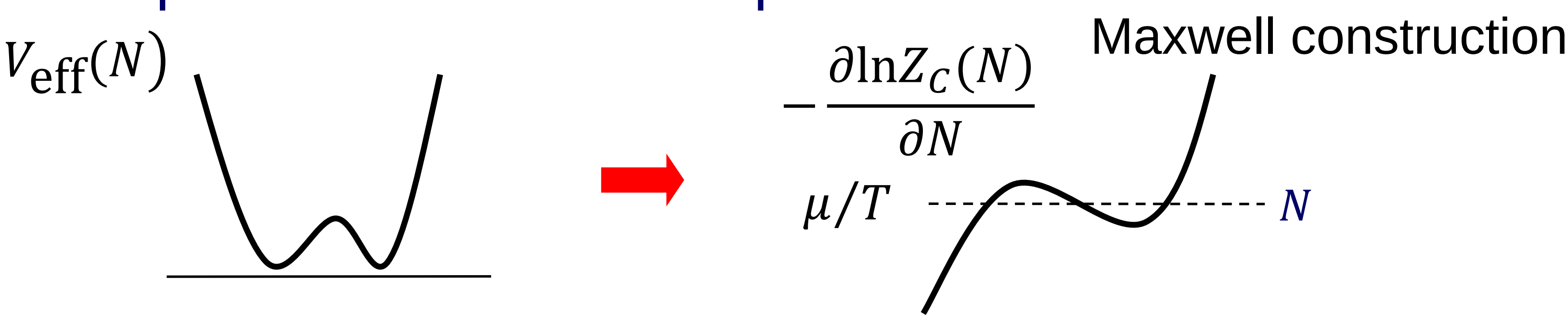
- Effective potential as a function of the quark number *N*.

$$V_{\mathrm{eff}}(N) = -\ln W(N) = -\ln Z_C(T,N) - N\frac{\mu}{T}$$

- At the minimum,

$$\frac{\partial V_{\mathrm{eff}}(N)}{\partial N} = -\frac{\partial \ln W(N)}{\partial N} = -\frac{\partial \ln Z_C(T,N)}{\partial N} - \frac{\mu}{T} = 0$$

- First order phase transition: Two phases coexist.



In the thermodynamic limit, $\frac{\mu}{T}(N) = -\frac{\partial \ln Z_C(T,N)}{\partial N}$

Hopping parameter expansion, Fugacity expansion

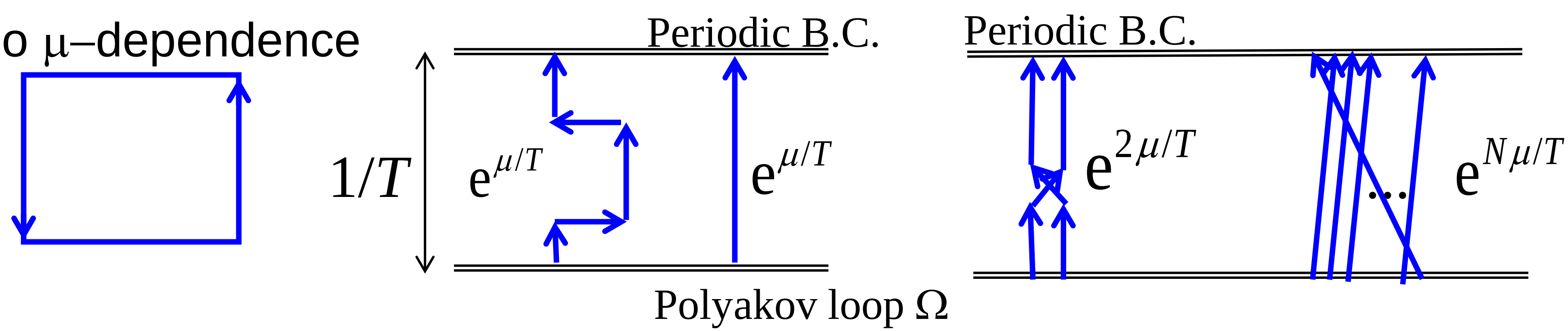
- Grand partition function $$Z = \int \prod_{x,\mu} dU_\mu(x)\ (\det M)^{N_f}\ e^{-S_g}$$
  Quark matrix
- Hopping parameter expansion [K~1/(quark mass)]

$$\ln(\det M(K)) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{\partial^n(\ln\det M)}{\partial K^n}\right]_{K=0} K^n = \sum_{n=1}^{\infty}\frac{1}{n!} D_n K^n$$

- D_n: Sum of all n-step Wilson loops (connected)

winding number N=0 N=1 N=2 N

No μ–dependence



- Classify the Wilson loops by the winding number.
- Fugacity expansion: expansion with the winding number N.

$$Z_{GC}(T,\mu) = \sum_N Z_C(N,T)\exp(N\mu/T)$$

Center symmetry

- Quenched QCD (no dynamical quarks, det*M*=1)
- Center of SU(3) group: $U_{\mathrm{center}} = \omega I,\ \omega = \{1,\ e^{\frac{2\pi i}{3}},\ e^{\frac{4\pi i}{3}}\}$
- On one time slice, $U_4 \Rightarrow \omega U_4$
- Polyakov loop Ω changes as $\langle \Omega \rangle \Rightarrow \omega \langle \Omega \rangle$
- $\langle \Omega \rangle = 0,$ when the symmetry is unbroken.
- winding number N loop

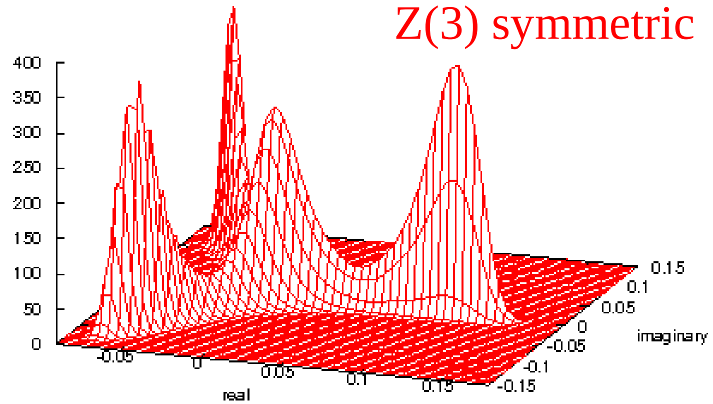
$$\langle \Omega_N \rangle \Rightarrow \omega^N \langle \Omega_N \rangle$$

- Canonical partition

$$Z_C(N,T) \Rightarrow \omega^N Z_C(N,T)$$

⇨ $Z_C(N,T) = 0,$ when $N \neq 3 \times (\mathrm{integer})$

Ω in the complex plane



Probability distribution at T_c

- For U(1) gage theory, $Z_C(N,T) \Rightarrow e^{iN\theta} Z_C(N,T) \quad (0 \leq \theta < 2\pi)$

⇨ $Z_C(N,T) = 0,$ for $N \neq 0$

We discuss U(1) gauge theory.

Center symmetry in U(1) gauge theory

- Centers of group are all members $U = e^{i\theta}$.
- Under the center transformation,

$$Z_{GC}(T,\mu) = \sum_N Z_C(N,T)\underline{e^{iN\theta}}e^{N\mu/T},$$

- Except for N=0, the canonical partition function is zero.

$$Z_{GC}(T,\mu) = \frac{1}{2\pi}\int\left[\sum_N Z_C(N,T)e^{iN\theta}e^{N\mu/T}\right]d\theta = Z_C(0,T)+0+0+\cdots$$

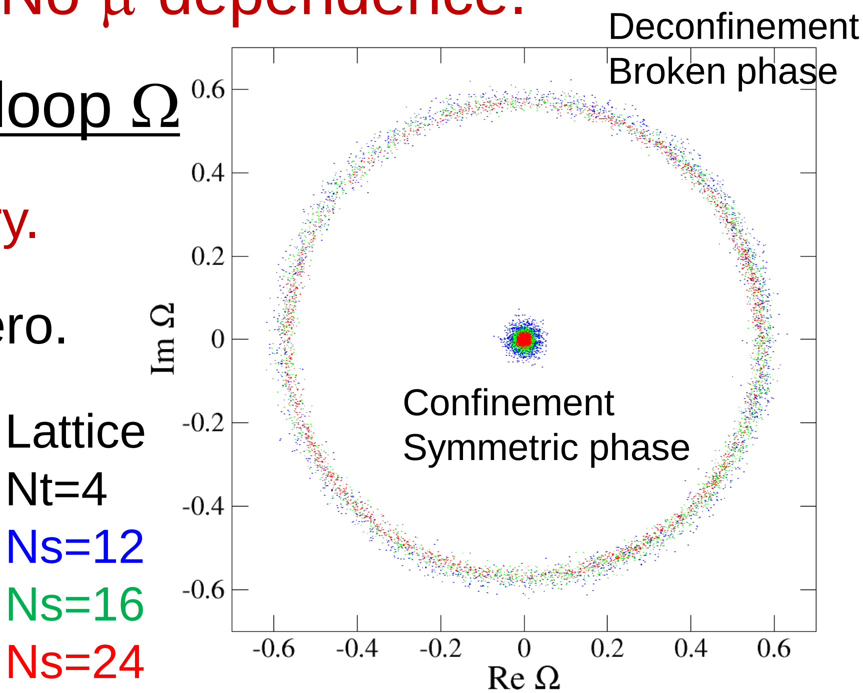
Charged particles cannot exist. No μ-dependence.

Probability distribution of Polyakov loop Ω

The distribution is always U(1) symmetry.

The expectation value of Ω is always zero.

To discuss the symmetry breaking,
Explicit breaking term: required,
e.g. dynamical quarks.



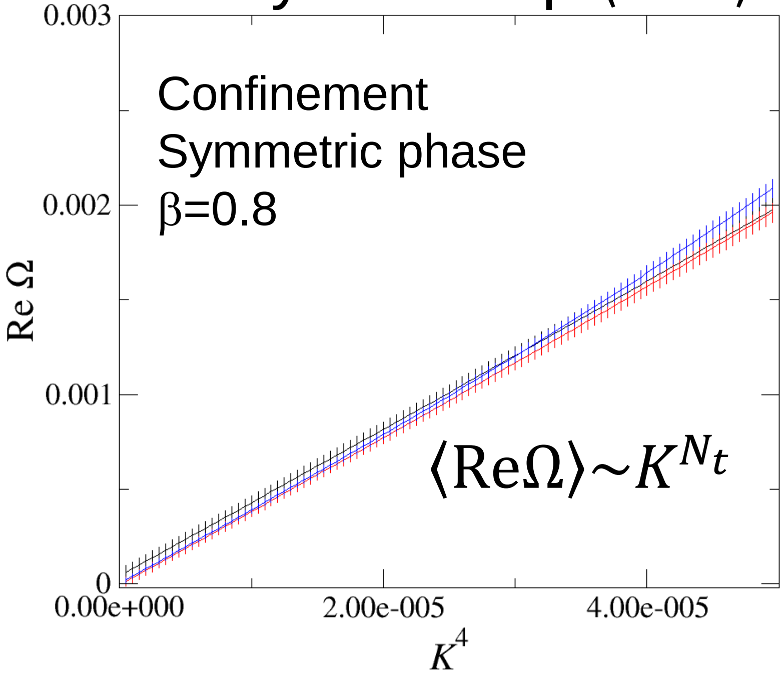
Spontaneous Symmetry Breaking
Adding dynamical quark with a small K

K~1/(quark mass)

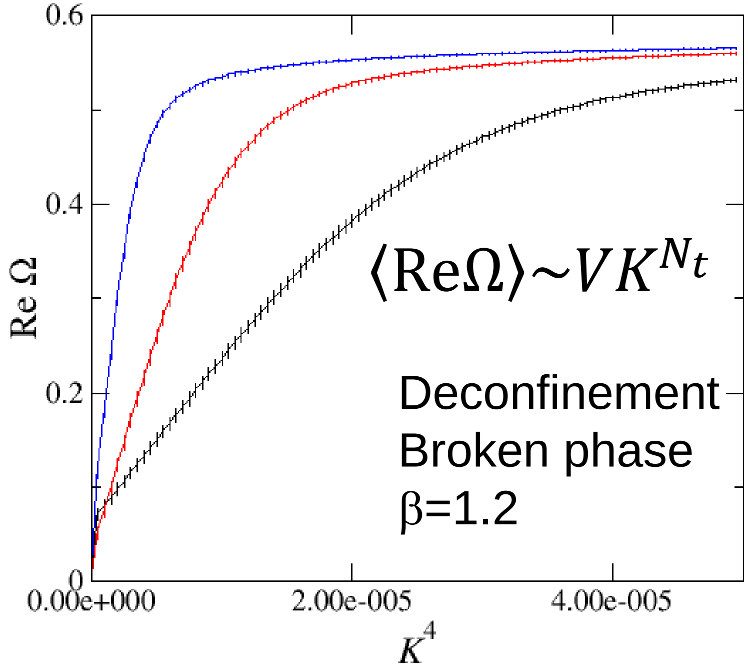
V: volume

- Center symmetry: explicitly broken.
- Double limit: $V \to \infty$ and $K \to 0$.

Polyakov loop $\langle \mathrm{Re}\Omega \rangle$



U(1) gauge theory



Lattice
Nt=4
Ns=24
Ns=16
Ns=12

In $V \to \infty$, $K = 0$ limit,

Symmetric phase: $\langle \mathrm{Re}\Omega \rangle$=0, Broken phase: $\langle \mathrm{Re}\Omega \rangle \sim V K^{N_t}$(finite)

Integrate over the complex phase θ

- Introducing the distribution function $W(|\Omega|)$

  <u>U(1) symmetry</u>

  $\rightarrow$ A function of only $|\Omega|$, <u>independent of θ</u>

$$\ln \det M = 6 \times 2^{N_t} N_s^3 K^{N_t} (\Omega + \Omega^*) + \cdots$$

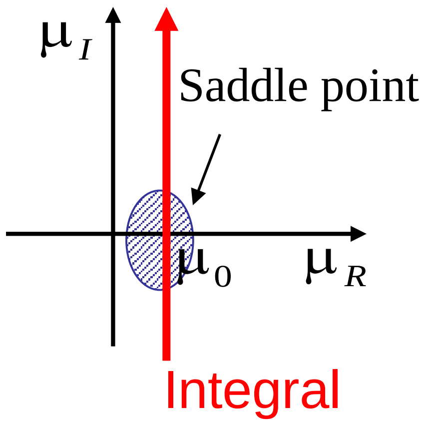
$$\langle \mathrm{Re}\Omega \rangle = \frac{1}{Z} \int DU \, \mathrm{Re}\Omega \, e^{\varepsilon V \mathrm{Re}\Omega} = \int |\Omega| \cos\theta \, e^{\varepsilon V |\Omega| \cos\theta} W(|\Omega|) \, d\theta \, d|\Omega|$$

$$= \varepsilon V \pi \int |\Omega|^2 \, W(|\Omega|) \, d|\Omega| + \cdots \qquad [\, \varepsilon \sim \text{◍} K^{N_t} \,]$$

$$\langle \mathrm{Re}\Omega^n \rangle = \frac{1}{Z} \int DU \, \mathrm{Re}\Omega^n e^{\varepsilon V \mathrm{Re}\Omega} = \frac{(\varepsilon V)^n \pi}{n! \, 2^{n-1}} \int |\Omega|^{2n} \, W(|\Omega|) d|\Omega| + \cdots$$

No complex phase, No sign problem

Canonical partition function by Saddle point approximation (S.E., Phys. Rev. D78, 074507 (2008))



- Inverse Laplace transformation

$$Z_C(T,N) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d(\mu_I/T)\, e^{-N(\mu_0/T+i\mu_I/T)} Z_{GC}(T,\mu_0+i\mu_I)$$

- Saddle point: z_0

$$D'(z_0) = \left(\frac{N_f}{V}\frac{\partial(\ln\det M)}{\partial(\mu/T)}\right)_{\frac{\mu}{T}=z_0} = \rho$$

Arbitrary μ_0

- Canonical partition function in a saddle point approximation

$$\frac{Z_C(T,\rho)}{Z_{\text{quench}}(T)} = \frac{1}{\sqrt{2\pi}}\left\langle \exp[V(D(z_0)-\rho z_0)]e^{-i\alpha/2}\sqrt{\frac{1}{V|D''(z_0)|}}\right\rangle_{\text{quench}} \equiv \frac{1}{\sqrt{2\pi}}\langle e^{F+i\varphi}\rangle_{\text{quench}}$$

$$D\left(\frac{\mu}{T}\right) = \frac{N_f}{V}(\ln\det M) \qquad D''\left(\frac{\mu}{T}\right) = \frac{N_f}{V}\frac{\partial^2(\ln\det M)}{\partial(\mu/T)^2} \equiv |D''|e^{i\alpha}$$

- Derivative of $\ln Z_C$

$$\frac{-1}{V}\frac{\partial \ln Z_C(T,\rho)}{\partial\rho} \approx \frac{\langle z_0\, e^{F+i\varphi}\rangle_{\text{quench}}}{\langle e^{F+i\varphi}\rangle_{\text{quench}}}$$

saddle point

reweighting factor

Similar to the reweighting method
(sign problem & overlap problem)

Heavy quark region (K: small) U(1) theory

Approximation: $\ln \det M \approx 6 \times 2^{N_t} N_s^3 K^{N_t} \left(e^{\mu/T} \Omega + e^{-\mu/T} \Omega^* \right)$

$$\frac{-1}{V} \frac{\partial \ln Z_C(T,\rho)}{\partial \rho} \approx \frac{\langle z_0\, e^{F+i\varphi} \rangle_{\text{quench}}}{\langle e^{F+i\varphi} \rangle_{\text{quench}}} \approx \frac{\varepsilon^N \int x_0\, e^{-V_{\text{eff}}} d|\Omega|}{\varepsilon^N \int e^{-V_{\text{eff}}} d|\Omega|}$$

$z_0 = x_0 + i y_0$

U(1) symmetric

$\varphi = -N\theta$ $\quad (\Omega = |\Omega| e^{i\theta})$

Phase of Ω

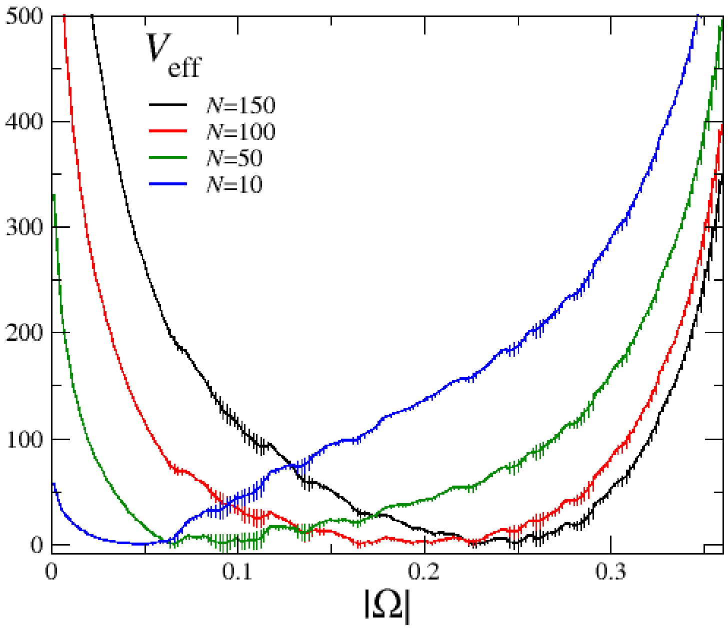
$$\langle e^{F+i\varphi} \rangle = \int e^F \cos(N\theta)\, e^{\varepsilon V |\Omega| \cos\theta} W(|\Omega|)\, d\theta\, d|\Omega|$$

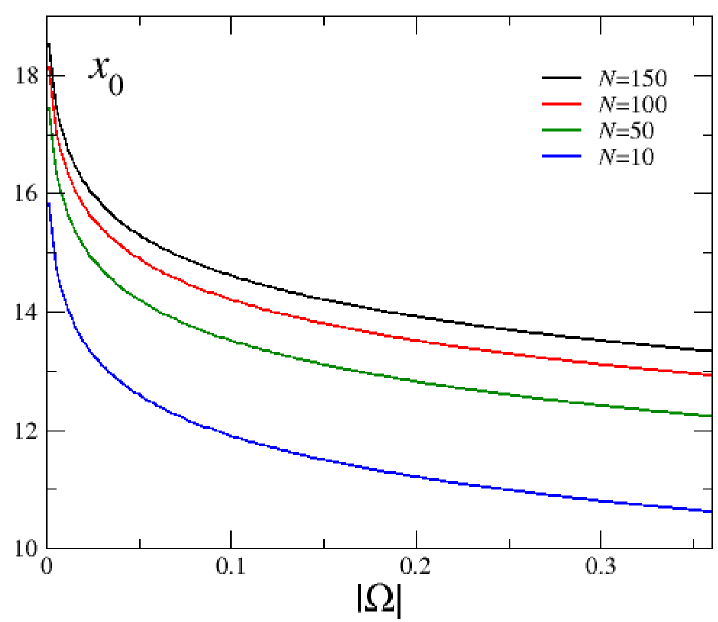
External source

$N = \rho V$

$(N_f = 2, K = 0.02)$ $\quad (24^3 \times 6$ lattice$)$

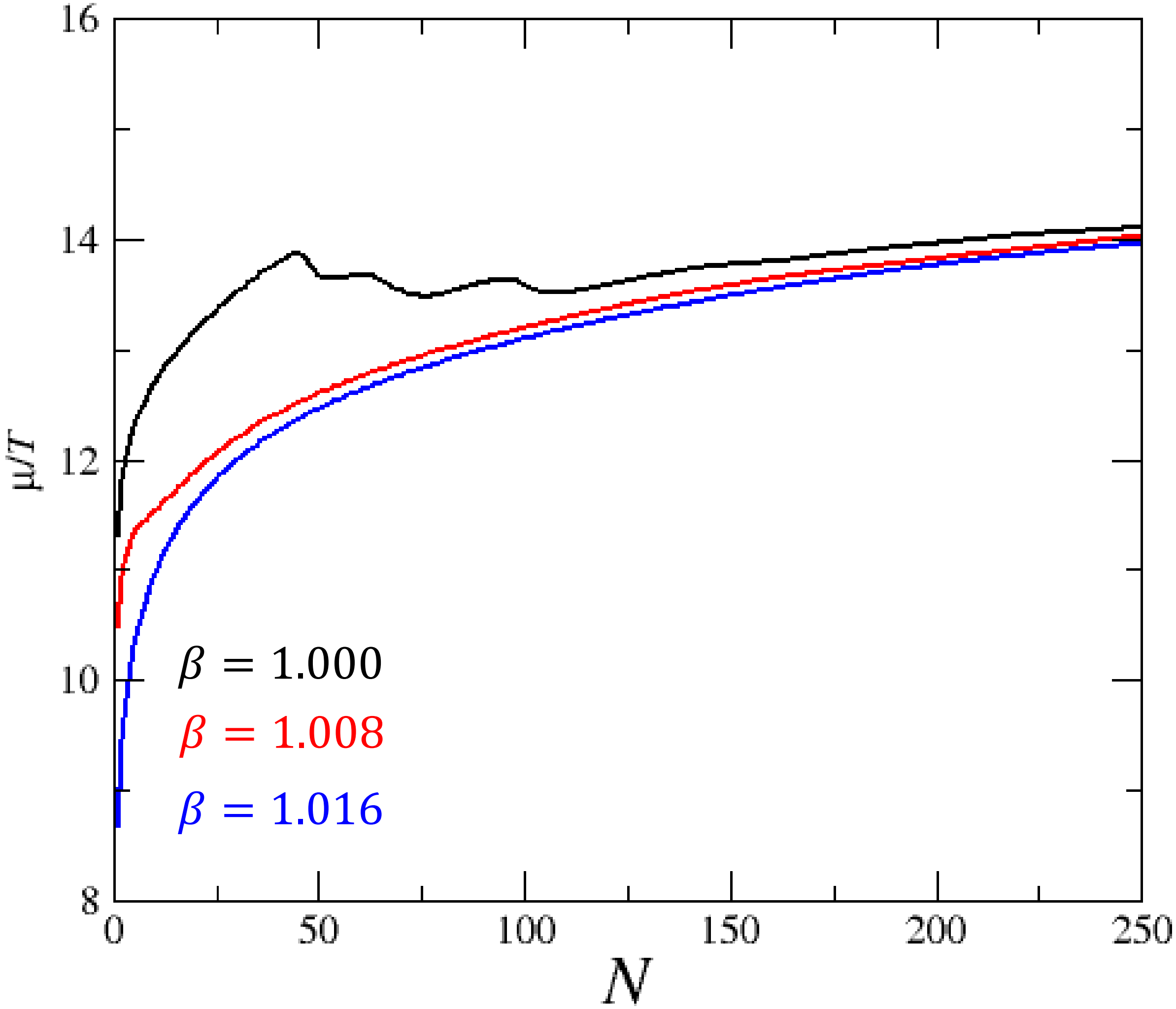
Solving the sign problem





Quark number N vs. $\frac{\mu}{T}$

- In the thermodynamic limit, $\frac{-1}{V}\frac{\partial \ln Z_C(T,\rho)}{\partial\rho} = \frac{\mu}{T}$



Critical β at μ=0

$\beta = 1.0096$

$(N_f = 2, K = 0.02)$

$(24^3 \times 6$ lattice$)$

$N = \rho V$

Application to SU(3) gauge theory

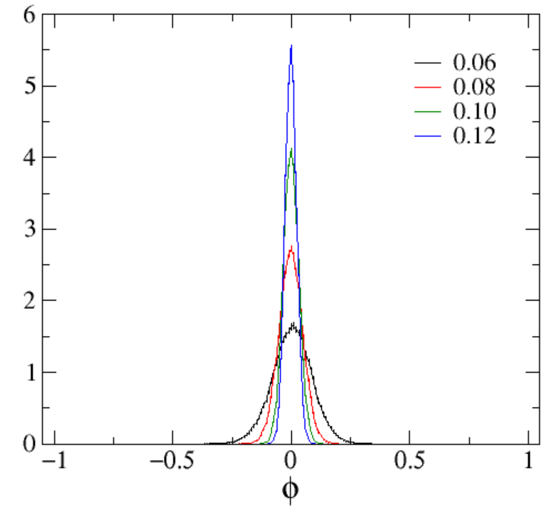
- SU(3) gauge theory in the low temperature phase:

  The Polyakov loop is U(1) symmetric (not Z(3)) in the large volume limit, if the Z(3) center symmetry is unbroken.

- In the high temperature phase of SU(3) gauge theory: The probability distribution of complex phase is well-approximated by a gaussian function.

Value of $|\Omega|$

Distribution of the phase Φ



Summary

- Quark number distribution function $W(N)$

$$Z_{GC}(T,\mu)=\sum_N Z_C(T,N)\exp(N\mu/T)\equiv\sum_N W(N)$$

- Considering the Center symmetry,
  - Canonical partition function of QCD is zero for $N\neq 3n\ \ (n\text{: integer})$.
  - For U(1) gauge theory, $Z_C=0$ for $N\neq 0$.
- It is important to break the center symmetry adding an external field term.
- Using the U(1) center symmetry, complex phase can be removed.
- For some cases, the sign problem is solved
  - U(1) gauge theory. Quarks are heavy.
  - Deep confinement phase
  - Deconfinement phase (the sign problem is not serious.)