# Towards complex Langevin simulations in superfluid phases

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# Application of Complex Langevin

Supersymmetric matrix model

Gauge theory with a θ term

QCD at finite density

Hubbard model away from half-filling

Bose gas in a rotating frame

Spin-orbit coupled bosons

Cold fermionic gasses

- with mass imbalance
- with population imbalance
- with three-body interaction

# Application of Complex Langevin

Supersymmetric matrix model

Gauge theory with a θ term

QCD at finite density — color superconductivity

Hubbard model away from half-filling — High-T superconductivity

Bose gas in a rotating frame

Spin-orbit coupled bosons

Cold fermionic gasses

- with mass imbalance — Sarma phase
- with population imbalance — FFLO superconductivity
- with three-body interaction

# Application of Complex Langevin

Superconductivity is ubiquitous !

But, conventional Monte Carlo method suffers from sign problem.

Q. Will Complex Langevin solve this problem ?

# Application of Complex Langevin

Superconductivity is ubiquitous !

But, conventional Monte Carlo method suffers from sign problem.

Q. Will Complex Langevin solve this problem ?

A. (Maybe) Yes !

But, be careful with the limitation of the Complex Langevin.

Aarts, Seiler, Stamatescu (2010), Aarts, James, Seiler, Stamatescu (2011),

Nishimura, Shimasaki (2015), Nagata, Nishimura, Shimasaki (2016)

# Singular drift problem

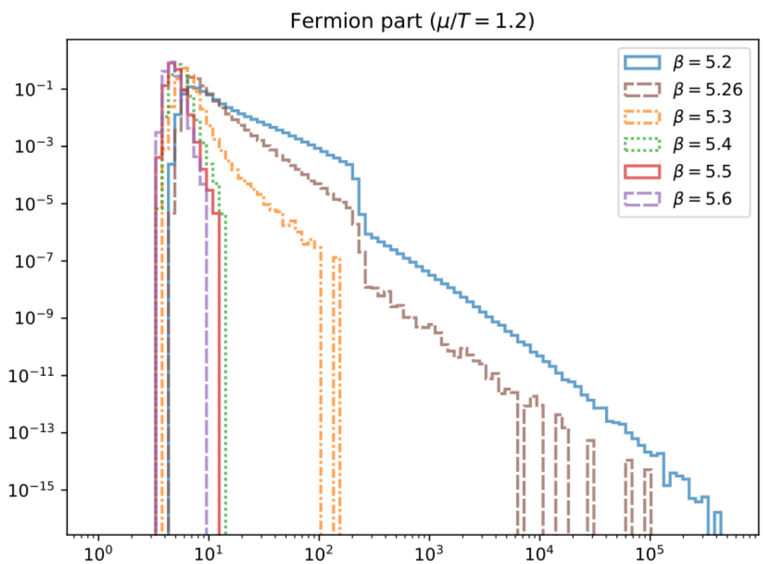
Nishimura, Shimasaki (2015)

Langevin equation:

Fermion determinant

$$\frac{d\phi}{dt} = -\frac{\partial(S(\phi) - \log \det M(\phi))}{\partial\phi} + \eta$$

$$\frac{\partial}{\partial\phi}\log\det M(\phi) \propto \frac{1}{\det M(\phi)}$$




This can be singular !

↓

Power law distribution

↓

Complax Langevin fails

# Banks-Casher (type) relation

Banks, Casher (1980), Kanazawa, Yamamoto (2016)

Splittorff (2016), Nagata, Nishimura, Shimasaki (2016)

Chiral condensate $\propto$ # of Dirac zero modes

Cooper pair condensate $\propto$ # of zero modes of $M(\phi)$

Inverse Nambu-Gor'kov

Green function

Assumptions:

- Thermodynamic limit
- Chiral limit (external source term goes to 0.)
- Density channel (auxiliary field $\phi$ ~ density profile)

# When complex Langevin fails ?

$$\frac{\partial}{\partial\phi}\log\det M(\phi) \propto \frac{1}{\det M(\phi)}$$

singularity of the drift term

~ zero modes exist

~ condensate exists

Exceptions:

- Finite system
- 1D, 2D
- Explicit breaking term
- Some tricky boundary conditions
- **Cooper channel**

# Partition function in Cooper channel

$$Z = \int \prod_\sigma \bar{\psi}_\sigma \psi_\sigma \int \mathcal{D}\bar{\Delta}\mathcal{D}\Delta \exp\left(-\int d\tau d^d x \left[\frac{1}{g}|\Delta|^2 - \bar{\Psi}\mathcal{G}^{-1}\Psi\right]\right)$$

$$\mathcal{G}^{-1} = \begin{pmatrix} G_\uparrow^{(p)-1} & \Delta \\ \bar{\Delta} & G_\downarrow^{(h)-1} \end{pmatrix},$$

$$G_\uparrow^{(p)-1} = -\partial_\tau + \frac{\nabla^2}{2m} + \mu,$$

$$G_\downarrow^{(h)-1} = -\partial_\tau - \frac{\nabla^2}{2m} - \mu$$

# Drift term in cooper channel

Then, at least within mean field approximation,

(and omitting σ dependence), the drift term is regular.

$$
\begin{aligned}
&\frac{\bar{\Delta}}{g} - \mathrm{Tr}\, G \frac{\delta^{-1}}{\delta \Delta} \\
=&\frac{\bar{\Delta}}{g} - \mathrm{Tr}\left[ \begin{pmatrix} G_{\uparrow}^{(p)-1} & \Delta \\ \bar{\Delta} & G_{\downarrow}^{(h)-1} \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right] \\
\simeq&\frac{\bar{\Delta}_0}{g} + \frac{T}{L^d} \sum_{n,p} \frac{1}{\omega_n^2 + \xi_p^2 + |\Delta_0|^2}
\end{aligned}
$$

We put $\Delta(\tau, x) = \Delta_0 = \text{const.}$ $\quad \xi_p = \frac{p^2}{2m} - \mu$

$\omega_n$ is Matsubara frequency.

# A proposal of lattice formulation

$$\det_{x,\tau}\left[\frac{\partial}{\partial\tau}+H\right] \propto \det_{x}\left[I+\exp\left(-\int_0^\beta d\tau H(\tau)\right)\right]$$

Based on

Blankenbecler, Scalapino, Sugar (1981)

$$\simeq \det_{x}\left[I+B_N\cdots B_2 B_1\right], \quad B_m = e^{-\Delta\tau K}e^{-\Delta\tau V_m}$$

$$K = \begin{pmatrix} -\frac{\nabla^2}{2m_\uparrow} & 0 \\ 0 & \frac{\nabla^2}{2m_\downarrow} \end{pmatrix}$$

$$V_m = \begin{pmatrix} -\mu_\uparrow & -\sqrt{g}\Delta(m\Delta\tau) \\ -\sqrt{g}\Delta(m\Delta\tau) & +\mu_\downarrow \end{pmatrix}$$

It is known that one particle Hamiltonian should be exponentiated to reduce lattice artifact.

# Summary

- If drift term is singular, this method will fail. This will happen in superfluid phases.

- If we use complex Langevin in the cooper channel, the drift term will be regular even in the superfluid phases.

- A lattice formulation in the cooper channel is proposed.

Complex Langevin is not applied in nuclear physics.

If you are interested in this technology, let's discuss !

# Physical meaning of zero modes 1/2

Add U(1) explicit breaking term to the BCS action:

$$Z(j) = \int \prod_\sigma \bar{\psi}_\sigma \psi_\sigma e^{-S[\bar{\psi}_\uparrow, \psi_\uparrow, \bar{\psi}_\downarrow, \psi_\downarrow] + j \int_0^\beta d\tau \int d^d r (\psi_\downarrow \psi_\uparrow + \bar{\psi}_\uparrow \bar{\psi}_\downarrow)}$$

Cooper pair condensate (order parameter of U(1) SSB):

$$I \equiv \lim_{j \to 0} \lim_{V \to \infty} \frac{1}{\beta V} \frac{d}{dj} \log Z(j)$$

# Physical meaning of zero modes 2/2

One can show Banks-Caser type relation:

$$I = 2\pi \int_0^\infty d\Lambda\, \Lambda R_0(\Lambda^2) \delta(\Lambda)$$

Eigenvalue distribution of $M(\phi) = \left(G_\uparrow^{-1} - \sqrt{g}\phi\right)\left(G_\downarrow^{-1} - \sqrt{g}\phi\right)$

Infinite number of zero modes ⇒ finite condensate

Kanazawa, Yamamoto, PRD 93 (2016) 016010