Charged analogs of the Efimov effect

Yusuke Nishida (Tokyo Tech)

International symposium on Clustering as a Window on the Hierarchical Structure of Quantum Systems (CLUSHIQ2022)

Oct.31 - Nov.3 (2022) @ Sendai

Plan of this talk

1. Introduction
   - Efimov effect & discrete scale invariance

2. Charged analog in non-relativistic system
   - Efimovian states in hydrogen molecular ion

Y. Nishida, Phys. Rev. A 105, L010802 (2022)

3. Charged analog in relativistic system
   - Atomic collapse resonances & vacuum polarization in graphene

Y. Nishida, Phys. Rev. B 90, 165414 (2014)

Y. Nishida, Phys. Rev. B 94, 085430 (2016)

Efimov effect & discrete scale inv.

Efimov effect

- ✓ 3 bosons
- ✓ 3 dimensions
- ✓ s-wave resonance

→ Infinite bound states with universal scaling

$$E_n \sim (22.7)^{-2n} E_0$$

V. Efimov, PLB (1970)

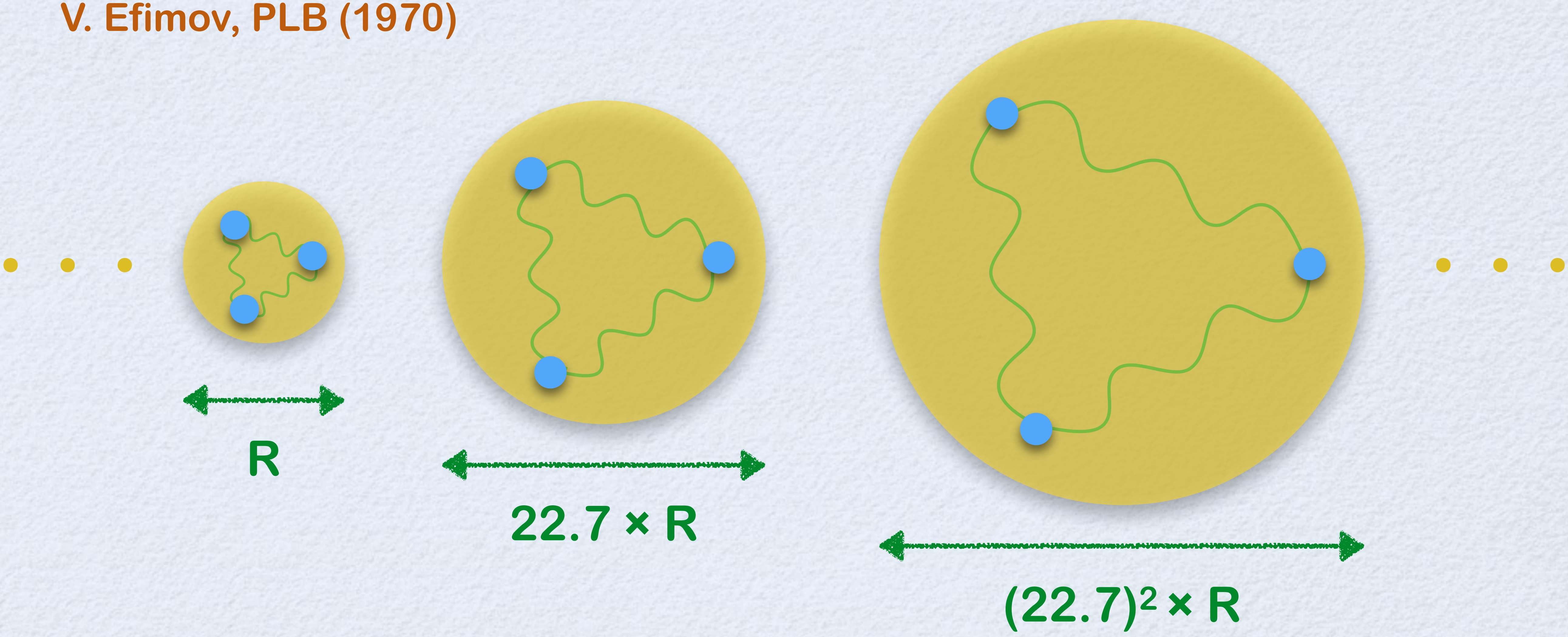
Efimov effect

- ✓ 3 bosons
- ✓ 3 dimensions
- ✓ s-wave resonance

→ Infinite bound states with universal scaling

$$E_n \sim (22.7)^{-2n} E_0$$

V. Efimov, PLB (1970)



Discrete scale invariance

Efimov effect

3-body Schrodinger equation

$$\left[T_1 + T_2 + T_3 + \underline{V_{12} + V_{23} + V_{31}}\right] \Psi(\vec{r}_1, \vec{r}_2, \vec{r}_3)$$
$$= E\Psi(\vec{r}_1, \vec{r}_2, \vec{r}_3)$$

Zero-range and infinite scattering length

Efimov effect

3-body Schrodinger equation

$$[T_1 + T_2 + T_3 + V_{12} + V_{23} + V_{31}]\,\Psi(\vec{r}_1, \vec{r}_2, \vec{r}_3) = E\Psi(\vec{r}_1, \vec{r}_2, \vec{r}_3)$$

Zero-range and infinite scattering length

Hyperradial motion

$$\left[-\frac{1}{2m}\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R}\right) + \frac{s^2}{2mR^2}\right]\psi(R) = -\frac{\kappa^2}{2m}\psi(R)$$

Scale invariant potential with s^2 = -1.013 < 0

induced by hyperangular motion

Efimov effect

3-body Schrodinger equation

$$[T_1 + T_2 + T_3 + V_{12} + V_{23} + V_{31}]\,\Psi(\vec{r}_1, \vec{r}_2, \vec{r}_3) = E\Psi(\vec{r}_1, \vec{r}_2, \vec{r}_3)$$

Zero-range and infinite scattering length

Hyperradial motion

$$\left[-\frac{1}{2m}\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R}\right) + \frac{s^2}{2mR^2}\right]\psi(R) = -\frac{\kappa^2}{2m}\psi(R)$$

Scale invariant potential with $s^2 = -1.013 < 0$

induced by hyperangular motion

Its solution $\psi(R) \propto K_{i|s|}(\kappa R)$

Efimov effect

Its solution $\psi(R) \propto K_{i|s|}(\kappa R)$

$\psi'/\psi|_{R=r_0}$ **has to be fixed by short-range B.C.**

Efimov effect

Its solution $\psi(R) \propto K_{i|s|}(\kappa R) \to \sin\left[|s| \ln\left(\kappa r_0\right) + \delta\right]$

$\psi'/\psi|_{R=r_0}$ **has to be fixed by short-range B.C.**

If $\kappa = \kappa_*$ **is a solution for** $\kappa r_0 \ll 1$**,**

$\kappa = (e^{\pi/|s|})^{-n}\kappa_*$ **are also solutions**

Efimov effect

Its solution $\psi(R) \propto K_{i|s|}(\kappa R) \to \sin\left[|s| \ln\left(\kappa r_0\right) + \delta\right]$

$\psi'/\psi|_{R=r_0}$ has to be fixed by short-range B.C.

If $\kappa = \kappa_*$ is a solution for $\kappa r_0 \ll 1$,

$\kappa = (e^{\pi/|s|})^{-n} \kappa_*$ are also solutions

$$\left[-\frac{1}{2m}\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R}\right) + \frac{s^2}{2mR^2}\right]\psi(R) = E\psi(R)$$

- Scale invariance is broken by short-range B.C. down to discrete scale invariance
- Long-range Coulomb potential is usually obstacle

Efimov effect

Its solution $\psi(R) \propto K_{i|s|}(\kappa R) \to \sin\left[|s| \ln(\kappa r_0) + \delta\right]$

$\psi'/\psi|_{R=r_0}$ has to be fixed by short-range B.C.

If $\kappa = \kappa_*$ is a solution for $\kappa r_0 \ll 1$,

$\kappa = (e^{\pi/|s|})^{-n}\kappa_*$ are also solutions

$$\left[-\frac{1}{2m}\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R}\right) + \frac{s^2}{2mR^2}\right]\psi(R) = E\psi(R)$$

- Scale invariance is broken by short-range B.C. down to discrete scale invariance
- Long-range Coulomb potential is usually obstacle

→ Discrete scale invariance for charged particles

Non-relativistic charged particles

L. D. Landau & E. M. Lifshitz, “Quantum Mechanics”

Hydrogen molecular ion

$V(R) \sim R^{-1}$

Hydrogen molecular ion

$V(R) \sim R^{-1} \Rightarrow R^{-2}$

Hydrogen molecular ion

V(R) ~ R^{-1} ⇒ R^{-2}

Born-Oppenheimer approximation (M≫m)

- Schrodinger equation for a light particle

$$\left[-\frac{\nabla_3^2}{2m}-\frac{kq^2}{|\vec{R}_1-\vec{r}_3|}-\frac{kq^2}{|\vec{R}_2-\vec{r}_3|}\right]\phi(\vec{r}_3)=\mathcal{E}_{\vec{R}_1\vec{R}_2}\phi(\vec{r}_3)$$

- Schrodinger equation for two heavy particles

$$\left[-\frac{\nabla_1^2}{2M}-\frac{\nabla_2^2}{2M}+\frac{kq^2}{|\vec{R}_1-\vec{R}_2|}+\mathcal{E}_{\vec{R}_1\vec{R}_2}\right]\Phi(\vec{R}_1,\vec{R}_2)$$

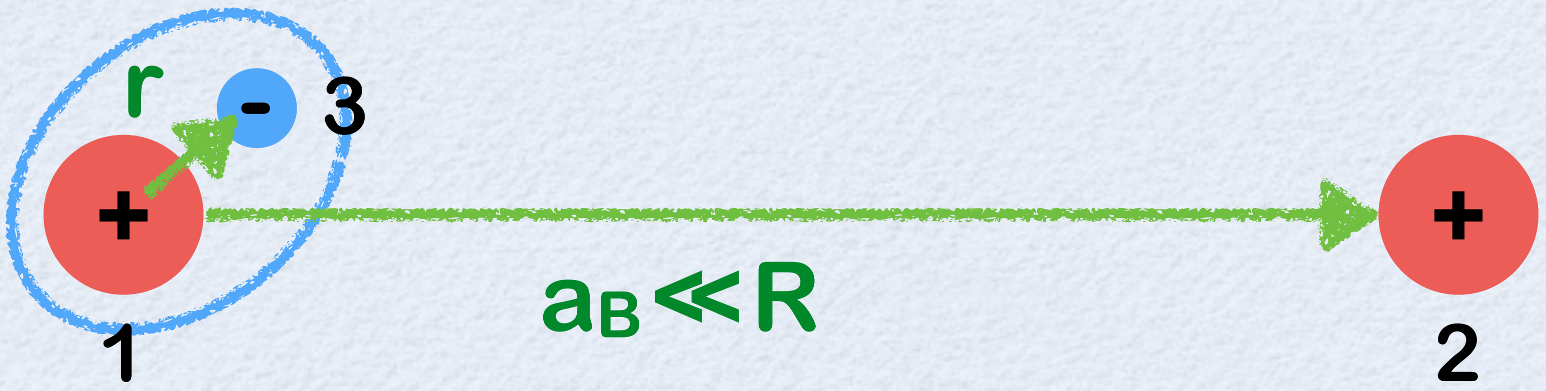
$$=E\Phi(\vec{R}_1,\vec{R}_2)$$

Hydrogen molecular ion

V(R) ~ R^{-1} ⇒ R^{-2}

Hydrogen-like atom $\mathcal{E}_n = -\dfrac{1}{2ma_B^2 n^2} \quad (n = 1, 2, \ldots)$

Hydrogen molecular ion



V(R) ~ R^{-1} ⇒ R^{-2}

Hydrogen-like atom $\mathcal{E}_n = -\frac{1}{2ma_B^2 n^2} \quad (n = 1, 2, \ldots)$

under electric field produced by far separated charge

$$V(\vec{r}) = \frac{kq^2}{R} - \frac{kq^2}{|\vec{R} - \vec{r}|} \simeq -\frac{kq^2}{R^2}\hat{R} \cdot \vec{r}$$

Hydrogen molecular ion

V(R) ~ R^{-1} ⇒ R^{-2}

Hydrogen-like atom $\mathcal{E}_n = -\frac{1}{2ma_B^2 n^2} \quad (n = 1, 2, \ldots)$

under electric field produced by far separated charge

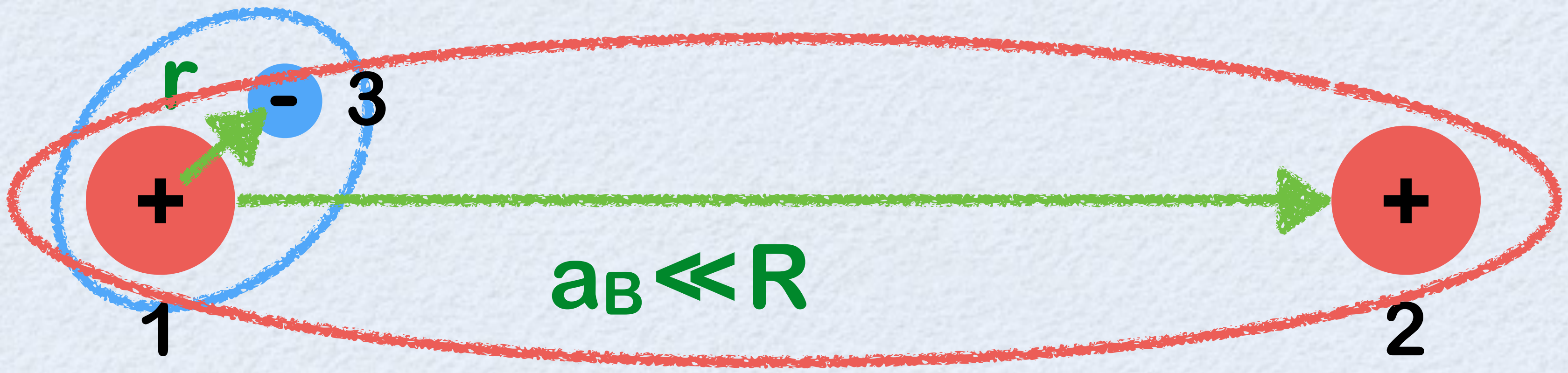
$$V(\vec{r}) = \frac{kq^2}{R} - \frac{kq^2}{|\vec{R} - \vec{r}|} \simeq -\frac{kq^2}{R^2}\hat{R}\cdot\vec{r}$$

1st-order perturbation $\Delta\mathcal{E}_n = \langle V(\vec{r})\rangle_n = 0 \quad (n = 1)$

$$\Delta\mathcal{E}_n = \langle V(\vec{r})\rangle_n = \pm\frac{3}{mR^2}, 0(\times 2) \quad (n = 2)$$

Scale invariant attraction for n=2,3,...

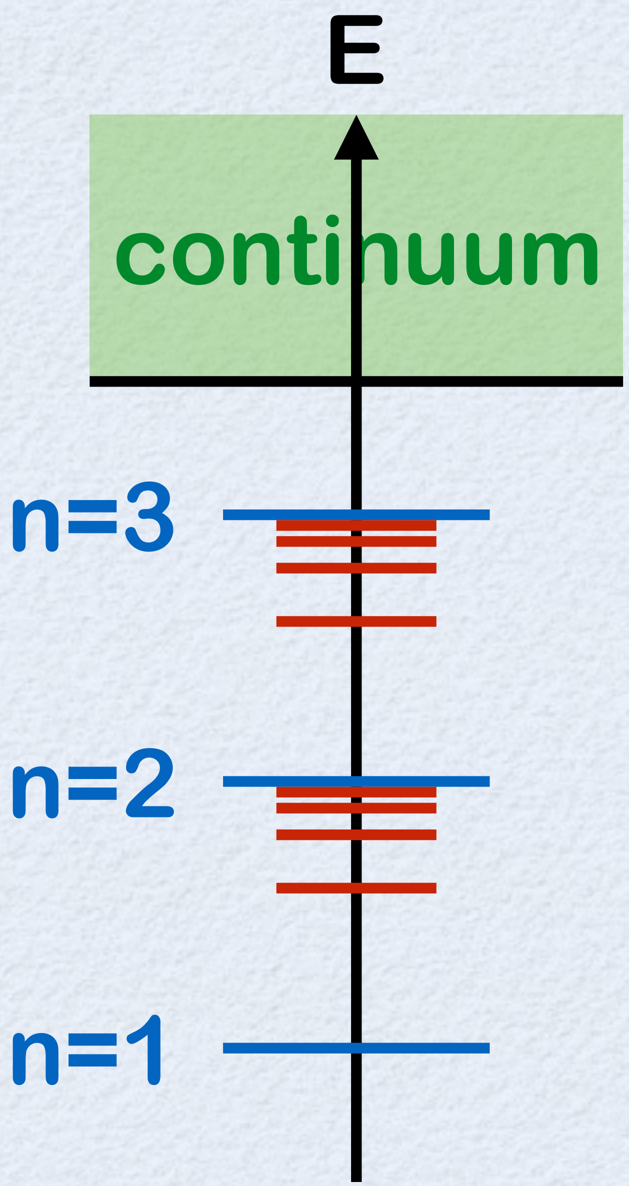
Hydrogen molecular ion



$V(R) \sim R^{-1} \Rightarrow R^{-2}$

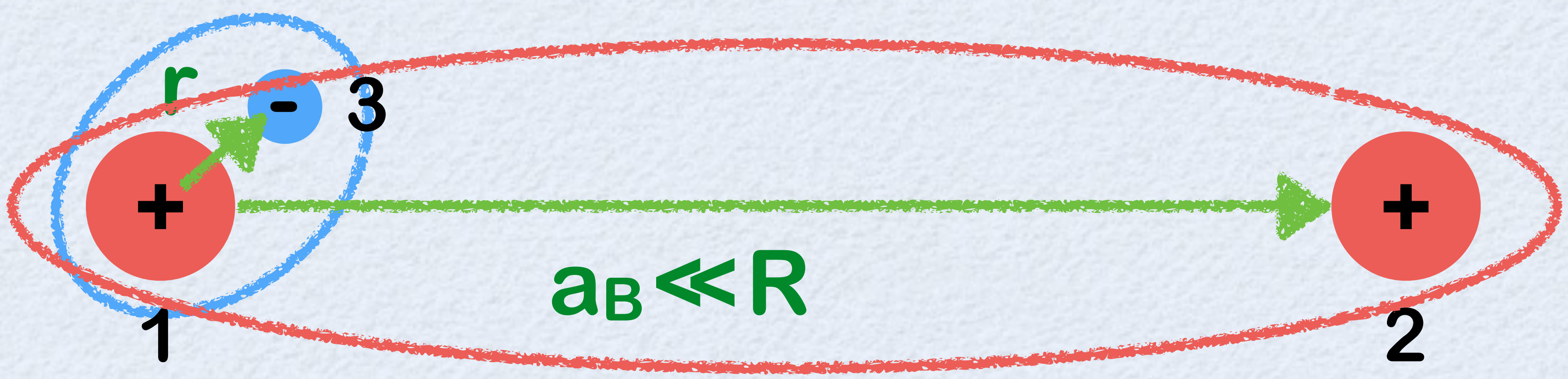
Infinite bound states obeying discrete scale invariance

toward thresholds of $(H)_{n=2,3,\ldots}$



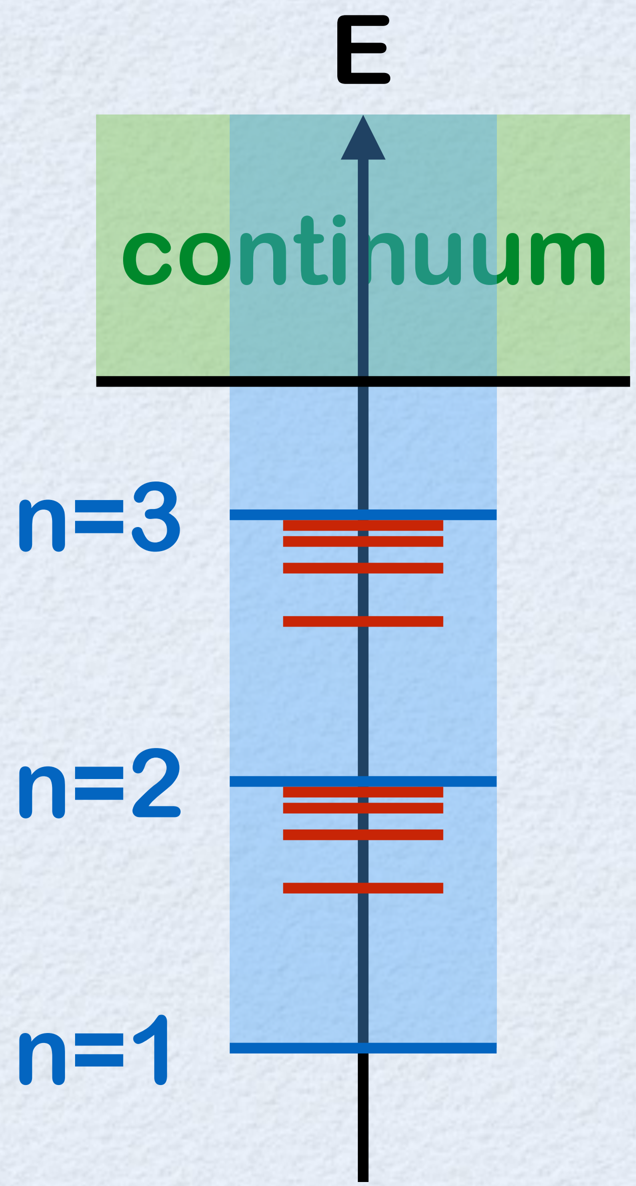
Y. Nishida, Phys. Rev. A 105, L010802 (2022)

Hydrogen molecular ion



$V(R) \sim R^{-1} \Rightarrow R^{-2}$

Infinite bound states obeying discrete scale invariance



toward thresholds of $(H)_{n=2,3,\ldots}$

- Efimovian states are resonances embedded into continuum of $(H)_{n=1}$
- Relevant to H_2^+ ions or trions (nuclear systems?)
- Future work: Their width and experimental probe

Y. Nishida, Phys. Rev. A 105, L010802 (2022)

Relativistic charged particles

Atomic collapse

Hydrogen-like atom from Dirac equation

$$\left[\vec{\alpha}\cdot\vec{p}+\beta m-\frac{Z\alpha}{r}\right]\psi(\vec{r})=E\psi(\vec{r})$$

Coulomb potential is scale invariant

$$\Rightarrow \quad E_{n'j}=\frac{m}{\sqrt{1+\frac{(Z\alpha)^2}{[n'+\sqrt{(j+\frac{1}{2})^2-(Z\alpha)^2}]^2}}}$$

Atomic collapse

Hydrogen-like atom from Dirac equation

$$\left[\vec{\alpha}\cdot\vec{p}+\beta m-\frac{Z\alpha}{r}\right]\psi(\vec{r})=E\psi(\vec{r})$$

Coulomb potential is scale invariant

$$E_{n'j}=\frac{m}{\sqrt{1+\frac{(Z\alpha)^2}{[n'+\sqrt{(j+\frac{1}{2})^2-(Z\alpha)^2}]^2}}}$$

becomes complex for $Z>\frac{j+\frac{1}{2}}{\alpha}>137$

"Atomic collapse" Y. B. Zeldovich & V. S. Popov (1971)

Atomic collapse

Hydrogen-like atom from Dirac equation

$$\left[\vec{\alpha}\cdot\vec{p}+\beta m-\frac{Z\alpha}{r}\right]\psi(\vec{r})=E\psi(\vec{r})$$

Coulomb potential is scale invariant

$$E_{n'j}=\frac{m}{\sqrt{1+\frac{(Z\alpha)^2}{[n'+\sqrt{(j+\frac{1}{2})^2-(Z\alpha)^2}]^2}}}$$

becomes complex for $Z>\frac{j+\frac{1}{2}}{\alpha}>137$

"Atomic collapse" Y. B. Zeldovich & V. S. Popov (1971)

$$\psi(\vec{r})\sim e^{\pm i\sqrt{(Z\alpha)^2-(j+\frac{1}{2})^2}\,\ln r}\qquad(r\ll a_B)$$

signals discrete scale invariance

Atomic collapse

Hydrogen-like atom from Dirac equation

$$\left[\vec{\alpha}\cdot\vec{p}+\beta m-\frac{Z\alpha}{r}\right]\psi(\vec{r})=E\psi(\vec{r})$$

Coulomb potential is scale invariant

- Z>137 is not yet achieved with a single nucleus but may be realized by colliding two heavy nuclei

W. Greiner, B. Muller & J. Rafelski, “Quantum Electrodynamics of Strong Fields”

Atomic collapse

Hydrogen-like atom from Dirac equation

$$\left[\vec{\alpha}\cdot\vec{p}+\beta m-\frac{Z\alpha}{r}\right]\psi(\vec{r})=E\psi(\vec{r})$$

Coulomb potential is scale invariant

- Z>137 is not yet achieved with a single nucleus but may be realized by colliding two heavy nuclei

W. Greiner, B. Muller & J. Rafelski, "Quantum Electrodynamics of Strong Fields"

- Because $v_F/c \sim O(0.01)$, $\alpha_{\mathrm{eff}} = \dfrac{ke^2}{\hbar\text{“}v_{\mathrm{F}}\text{”}} \sim O(1)$

"superheavy nucleus" can be realized by a charged impurity with Z~O(1) on graphene

V. M. Pereira, J. Nilsson & A. H. Castro Neto, PRL (2007)

A. V. Shytov, M. I. Katsnelson & L. S. Levitov, PRL (2007)

Graphene

2D massless Dirac equation with a charged impurity

$$\left[\vec{\sigma}\cdot\vec{p} - \frac{Z\alpha_{\text{eff}}}{r}\right]\psi(\vec{r}) = E\psi(\vec{r})$$

Scale invariance

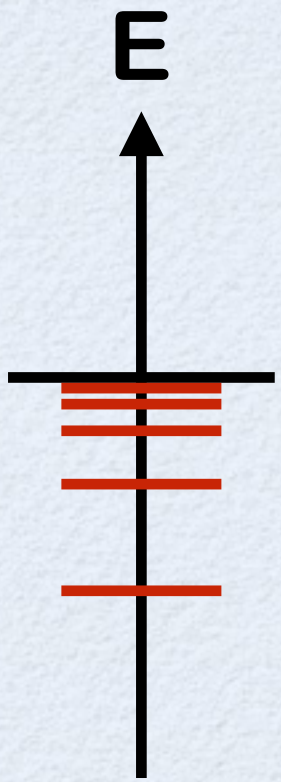
Graphene

2D massless Dirac equation with a charged impurity

$$\left[\vec{\sigma}\cdot\vec{p} - \frac{Z\alpha_{\text{eff}}}{r}\right]\psi(\vec{r}) = E\psi(\vec{r})$$

Scale invariance is broken by short-range B.C.

down to discrete scale invariance for $Za_{eff}>1$



Infinite bound states

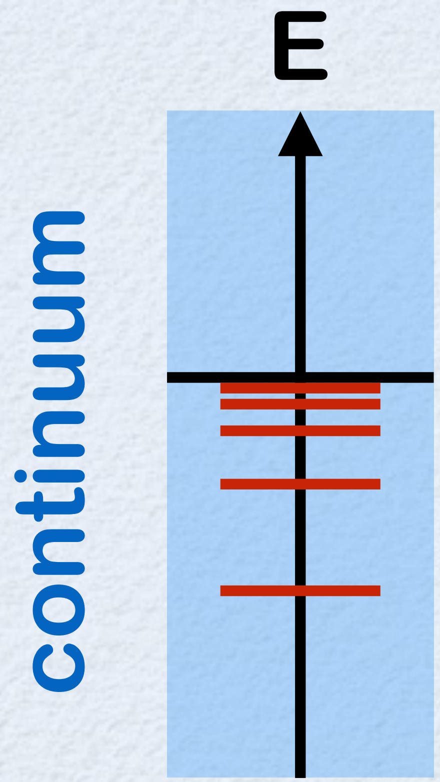
Graphene

2D massless Dirac equation with a charged impurity

$$\left[\vec{\sigma}\cdot\vec{p} - \frac{Z\alpha_{\text{eff}}}{r}\right]\psi(\vec{r}) = E\psi(\vec{r})$$

Scale invariance is broken by short-range B.C.

down to discrete scale invariance for $Z\alpha_{\text{eff}}>1$



Infinite ~~bound states~~ resonances

"Atomic collapse resonances"

⇒ DoS peaks probed by STM

Graphene

Scale invariance is broken by short-range B.C.

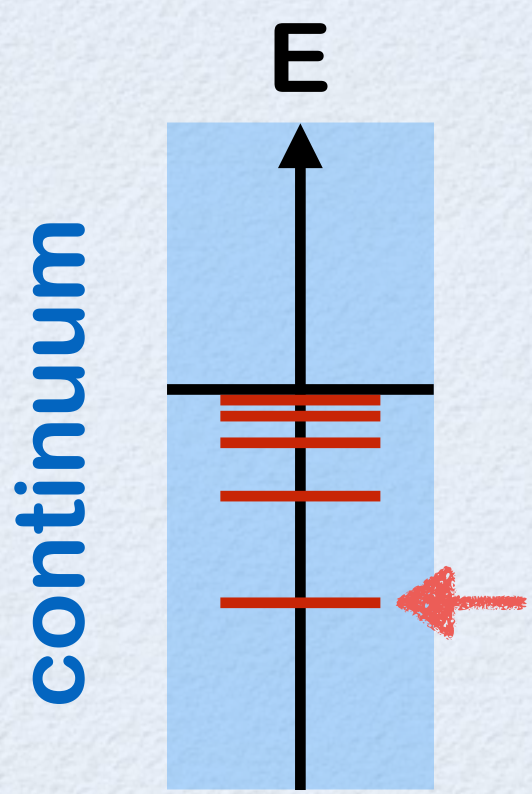
down to **discrete scale invariance** for $Z\alpha_{eff}>1$

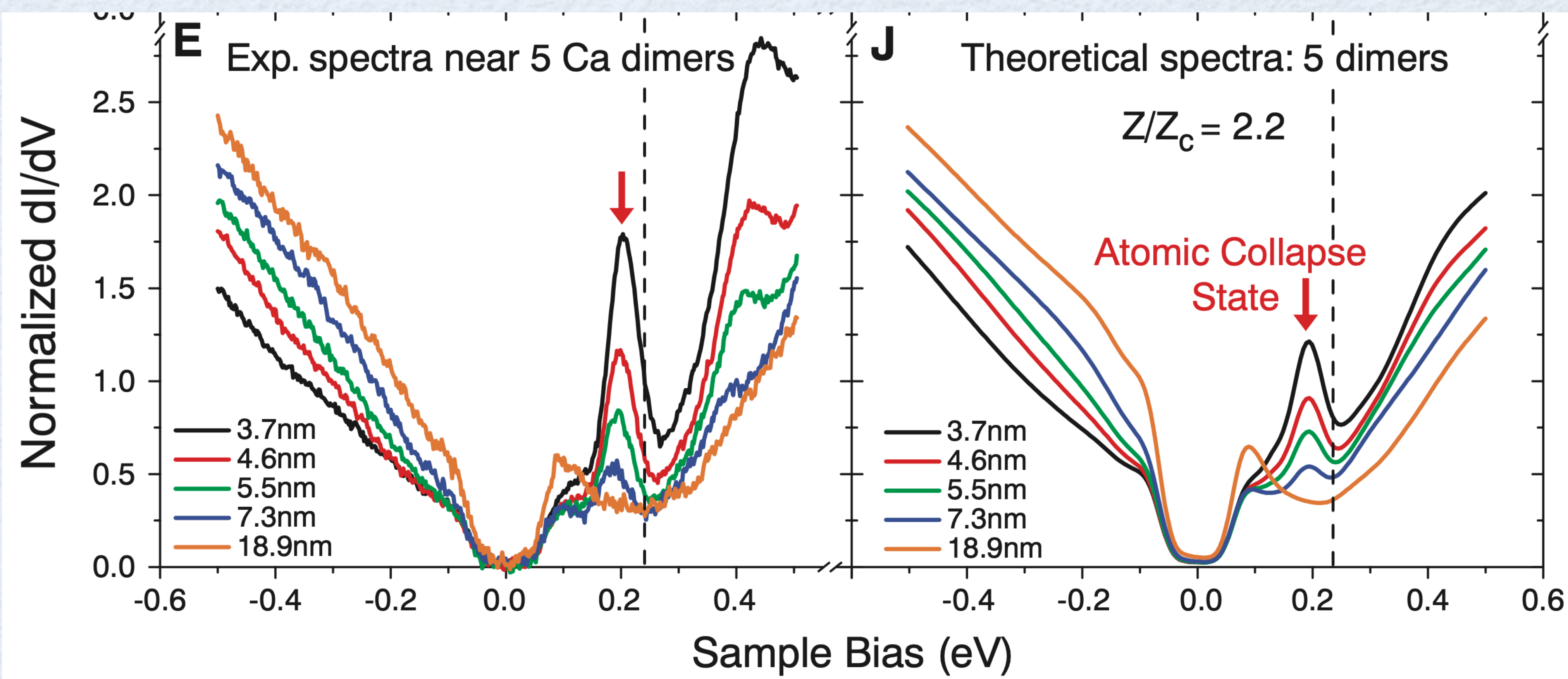
Infinite ~~bound states~~ resonances

"Atomic collapse resonances"

⇒ DoS peaks probed by STM

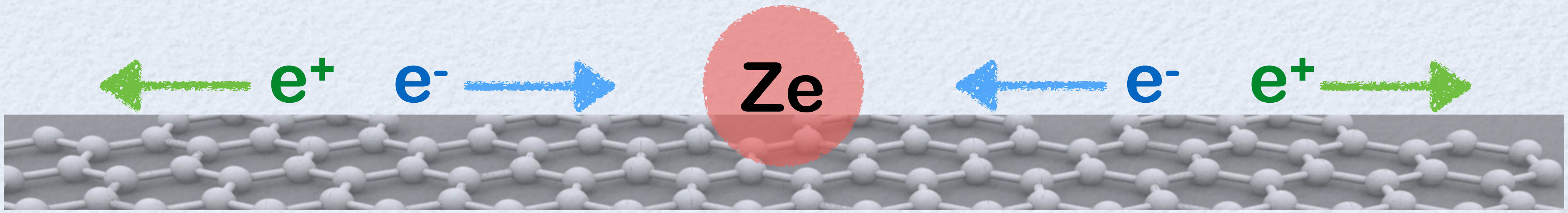
M. F. Crommie et al.
Science 340, 734 (2013)





Cond-mat realization of "superheavy nucleus"

Vacuum polarization



Y. Nishida, Phys. Rev. B 90, 165414 (2014); Phys. Rev. B 94, 085430 (2016)

Vacuum polarization

Charge distribution of electrons $n(r) = \sum_{E<0} |\psi_E(\vec{r})|^2$

Y. Nishida, Phys. Rev. B 90, 165414 (2014); Phys. Rev. B 94, 085430 (2016)

Vacuum polarization

Charge distribution of electrons $n(r) = \sum_{E<0} |\psi_E(\vec{r})|^2$

- Scale invariance ⇒ $n(r) = \dfrac{C}{r^2}$ Power law

Y. Nishida, Phys. Rev. B 90, 165414 (2014); Phys. Rev. B 94, 085430 (2016)

Vacuum polarization

Charge distribution of electrons $n(r) = \sum_{E<0} |\psi_E(\vec{r})|^2$

- Scale invariance ⇒ $n(r) = \dfrac{C}{r^2}$ **Power law**
- Discrete scale invariance ⇒ $n(r) = \dfrac{F(\ln r)}{r^2}$

Power law + log-periodic oscillation

Y. Nishida, Phys. Rev. B 90, 165414 (2014); Phys. Rev. B 94, 085430 (2016)

Vacuum polarization

Charge distribution of electrons $n(r) = \sum_{E<0} |\psi_E(\vec{r})|^2$

- Scale invariance ⇒ $n(r) = \dfrac{C}{r^2}$ Power law
- Discrete scale invariance ⇒ $n(r) = \dfrac{F(\ln r)}{r^2}$

Power law + log-periodic oscillation

$$F_j(r/r_j^*) = \frac{\gamma}{2\pi^2}\,\mathrm{Re}\int_0^\infty dz\,\frac{\Gamma(1-ig+i\gamma)\Gamma(1-ig-i\gamma)}{\Gamma(1+2i\gamma)\Gamma(1-2i\gamma)}\left[\frac{1+\frac{(j-ig+i\gamma)\Gamma(1+2i\gamma)\Gamma(1-ig-i\gamma)}{(j-ig-i\gamma)\Gamma(1-2i\gamma)\Gamma(1-ig+i\gamma)}\left(\frac{r}{zr_j^*}\right)^{2i\gamma}}{1-\frac{(j-ig+i\gamma)\Gamma(1+2i\gamma)\Gamma(1-ig-i\gamma)}{(j-ig-i\gamma)\Gamma(1-2i\gamma)\Gamma(1-ig+i\gamma)}\left(\frac{r}{zr_j^*}\right)^{2i\gamma}}\right]$$
$$\times\, e^{-z}U(-ig+i\gamma,1+2i\gamma,z)U(1-ig-i\gamma,1-2i\gamma,z)$$

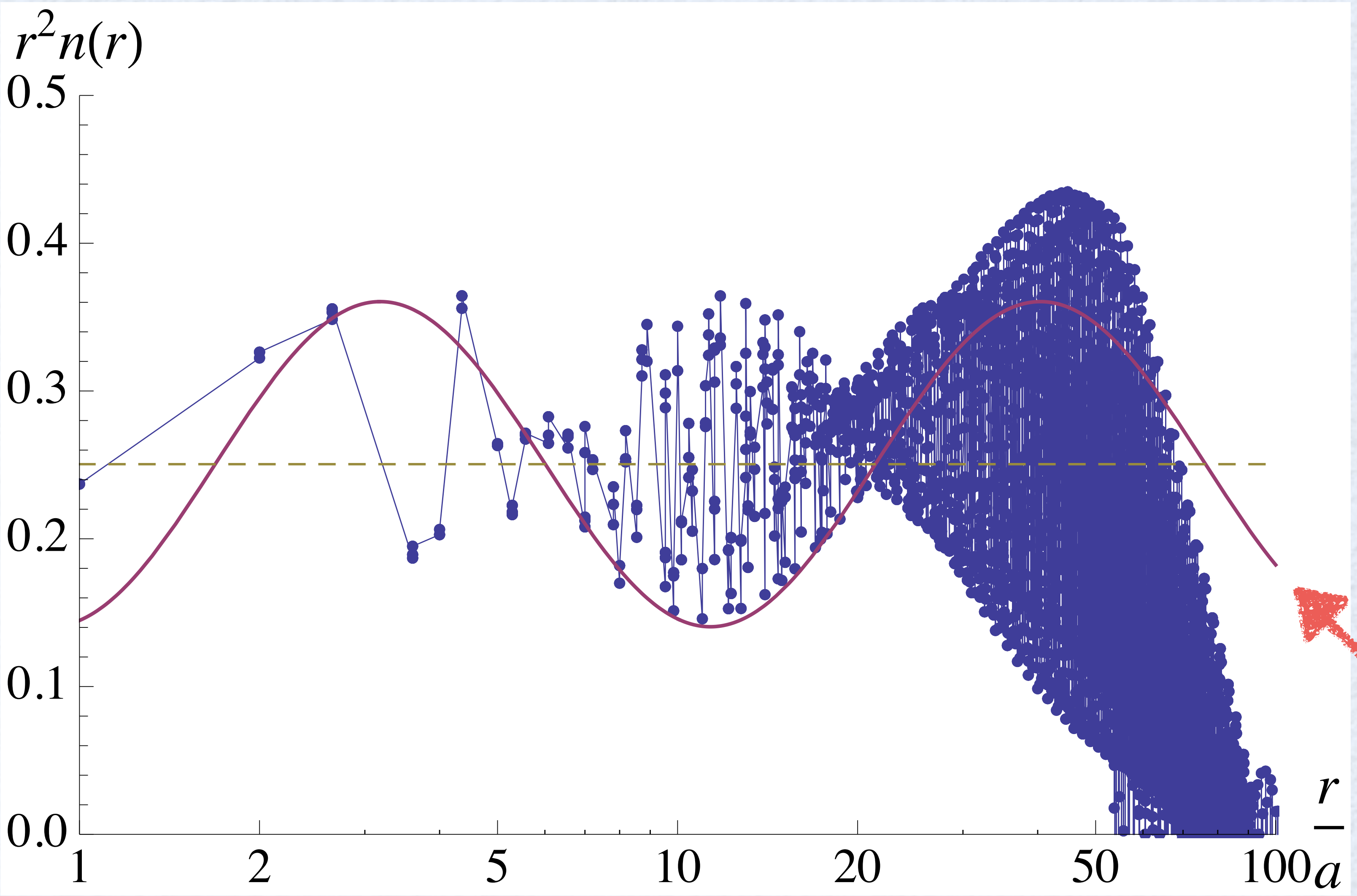
Y. Nishida, Phys. Rev. B 90, 165414 (2014); Phys. Rev. B 94, 085430 (2016)

Vacuum polarization

Comparison to lattice data for $Za_{eff}=4/3$

(exact diagonalization on honeycomb lattice with 124x124 sites)

V. M. Pereira, J. Nilsson & A. H. Castro Neto, PRL (2007)



Shape & period are predictions

Amplitude & phase are fit parameters

$$n(r) = \frac{F(\ln r)}{r^2}$$

- Envelop fits well our prediction but fast oscillation exists with its origin unknown

Summary and problems

1. Introduction
   - Efimov effect & discrete scale invariance

2. Charged analog in non-relativistic system
   - Efimovian states in hydrogen molecular ion

   ⇒ Their width and experimental probe?

3. Charged analog in relativistic system
   - Atomic collapse resonances & vacuum polarization in graphene

   ⇒ Possibility in heavy-ion collisions?

Suggestions are appreciated!